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Toric duality as Seiberg duality and brane diamonds*

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Abstract: We use field theory and brane diamond techniques to demonstrate that Toric Duality is Seiberg duality for $\mathcal{N} = 1$ theories with toric moduli spaces. This resolves the puzzle concerning the physical meaning of Toric Duality as proposed in our earlier work. Furthermore, using this strong connection we arrive at three new phases which can not be thus far obtained by the so-called "Inverse Algorithm" applied to partial resolution of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. The standing proposals of Seiberg duality as diamond duality in the work by Aganagic-Karch-Lüst-Miemiec are strongly supported and new diamond configurations for these singularities are obtained as a byproduct. We also make some remarks about the relationships between Seiberg duality and Picard-Lefschetz monodromy.

Keywords: Duality in Gauge Field Theories, Conformal Field Models in String Theory, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

Witten's gauged linear sigma approach [1] to $\mathcal{N} = 2$ super-conformal theories has provided deep insight not only to the study of the phases of the field theory but also to the understanding of the mathematics of Geometric Invariant Theory quotients in toric geometry. Thereafter, the method was readily applied to the study of the $\mathcal{N} = 1$ supersymmetric gauge theories on D-branes at singularities [2, 3, 4, 5]. Indeed the classical moduli space of the gauge theory corresponds precisely to the spacetime which the D-brane probes transversely. In light of this therefore, toric geometry has been widely used in the study of the moduli space of vacua of the gauge theory living on D-brane probes.

The method of encoding the gauge theory data into the moduli data, or more specifically, the F-term and D-term information into the toric diagram of the algebraic variety describing the moduli space, has been well-established [2, 3]. The reverse, of determining the SUSY gauge theory data in terms of a given toric singularity upon which the D-brane probes, has also been addressed using the method partial resolutions of abelian quotient singularities. Namely, a general non-orbifold singularity is regarded as a partial resolution of a worse, but orbifold, singularity. This "Inverse Procedure" was formalised into a linear optimisation algorithm, easily implementable on computer, by [6], and was subsequently checked extensively in [7].

One feature of the Inverse Algorithm is its non-uniqueness, viz., that for a given toric singularity, one could in theory construct countless gauge theories. This means that there are classes of gauge theories which have identical toric moduli space in the IR. Such a salient feature was dubbed in [6] as *toric duality*. Indeed in a follow-up work, [8] attempted to analyse this duality in detail, concentrating in particular on a method of fabricating dual theories which are physical, in the sense that they can be realised as world-volume theories on D-branes. Henceforth, we shall adhere to this more restricted meaning of toric duality.

Because the details of this method will be clear in later examples we shall not delve into the specifics here, nor shall we devote too much space reviewing the algorithm. Let us highlight the key points. The gauge theory data of D-branes probing abelian orbifolds is well-known (see e.g. the appendix of [8]); also any toric diagram can be embedded into that of such an orbifold (in particular any toric local Calabi-Yau threefold D can be embedded into $\mathbb{C}^3/(\mathbb{Z}_n \times \mathbb{Z}_n)$ for sufficiently large n. We can then obtain the subsector of orbifold theory that corresponds the gauge theory constructed for D. This is the method of "Partial Resolution."

A key point of [8] was the application of the well-known mathematical fact that the toric diagram D of any toric variety has an inherent ambiguity in its definition: namely any unimodular transformation on the lattice on which D is defined must leave D invariant. In other words, for threefolds defined in the standard lattice $\mathbb{Z}^3$, any $SL(3;\mathbb{Z})$ transformation on the vector endpoints of the defining toric diagram gives the same toric variety. Their embedding into the diagram of a fixed abelian orbifold on the other hand, certainly is different. Ergo, the gauge theory data one obtains in general are vastly different, even though per constructio, they have the same toric moduli space.

What then is this "toric duality"? How clearly it is defined mathematically and yet how illusive it is as a physical phenomenon. The purpose of the present writing is to make the first leap toward answering this question. In particular, we shall show, using brane setups, and especially brane diamonds, that known cases for toric duality are actually interesting realisations of Seiberg Duality. Therefore the mathematical equivalence of moduli spaces for different quiver gauge theories is related to a real physical equivalence of the gauge theories in the far infrared.

The paper is organised as follows. In section 2, we begin with an illustrative example of two torically dual cases of a generalised conifold. These are well-known to be Seiberg dual theories as seen from brane setups. Thereby we are motivated to conjecture in section 3 that toric duality is Seiberg duality. We proceed to check this proposal in section 4 with all the known cases of torically dual theories and have successfully shown that the phases of the partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ constructed in [6] are indeed Seiberg dual from a field theory analysis. Then in section 6 we re-analyse these examples from the perspective of brane diamond configurations and once again obtain strong support of the statement. From rules used in the diamond dualisation, we extracted a so-called "quiver duality" which explicits Seiberg duality as a transformation on the matter adjacency matrices. Using these rules we are able to extract more phases of theories not yet obtained from the Inverse Algorithm. In a more geometrical vein, in section 7, we remark the connection between Seiberg duality and Picard-Lefschetz and point out cases where the two phenomena may differ. Finally we finish with conclusions and prospects in section 8.

While this manuscript is about to be released, we became aware of the nice work [9], which discusses similar issues.

2. An illustrative example

We begin with an illustrative example that will demonstrate how Seiberg Duality is realised as toric duality.

2.1 The brane setup

The example is the well-known generalized conifold described as the hypersurface $xy = z^2w^2$ in $\mathbb{C}^4$, and which can be obtained as a $\mathbb{Z}_2$ quotient of the famous conifold $xy = zw$ by the action $z \to -z$, $w \to -w$. The gauge theory on the D-brane sitting at such a singularity can be established by orbifolding the conifold gauge theory in [10], as in [11]. Also, it can be derived by another method alternative to the Inverse Algorithm, namely performing a T-duality to a brane setup with NS-branes and D4-branes [11, 12]. Therefore this theory serves as an excellent check on our methods.

The setup involves stretching D4 branes (spanning 01236) between 2 pairs of NS and NS′ branes (spanning 012345 and 012389, respectively), with x^6 parameterizing a circle. These configurations are analogous to those in [13]. There are in fact two inequivalent brane setups (a) and (b) (see figure 1), differing in the way the NS- and NS′-branes are ordered in the circle coordinate.

Using standard rules [14, 13], we see from the figure that there are 4 product gauge groups (in the abelian case, it is simply $\mathrm{U}(1)^4$. As for the matter content, theory (a) has 8 bi-fundamental chiral multiplets A_i, B_i, C_i, D_i $i = 1, 2$ (with charge $(+1, -1)$ and $(-1, +1)$ with respect to adjacent U(1) factors) and 2 adjoint chiral multiplets $\phi_{1,2}$ as indicated. On the other hand (b) has only 8 bi-fundamentals, with charges as above. The superpotentials are respectively [15, 11]

$$(a) \quad W_a = -A_1A_2B_1B_2 + B_1B_2\phi_2 - C_1C_2\phi_2 + C_1C_2D_1D_2 - D_1D_2\phi_1 + A_1A_2\phi_1\,,$$
$$(b) \quad W_b = A_1A_2B_1B_2 - B_1B_2C_1C_2 + C_1C_2D_1D_2 - D_1D_2A_1A_2\,.$$

With some foresight, for comparison with the results later, we rewrite them as

$$W_a = (B_1B_2 - C_1C_2)(\phi_2 - A_1A_2) + (A_1A_2 - D_1D_2)(\phi_1 - C_1C_2) \tag{2.1}$$
$$W_b = (A_1A_2 - C_1C_2)(B_1B_2 - D_1D_2)\,. \tag{2.2}$$

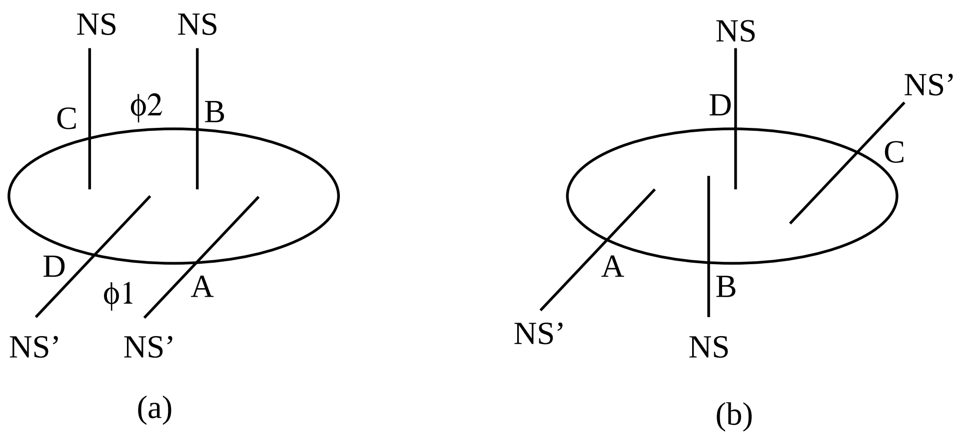

Figure 1: The two possible brane setups for the generalized conifold $xy = z^2w^2$. They are related to each other passing one NS-brane through an NS'-brane. A_i, B_i, C_i, D_i $i = 1, 2$ are bifundamentals while ϕ_1, ϕ_2 are two adjoint fields.

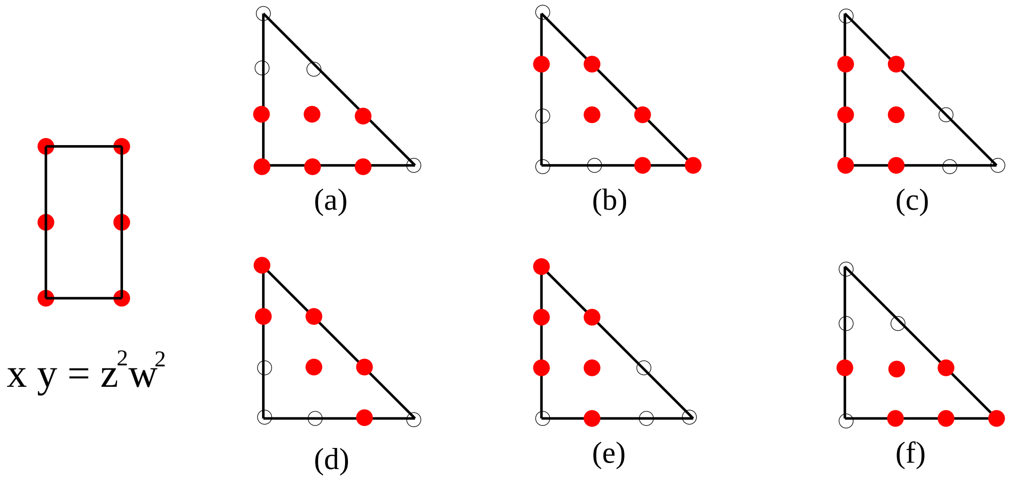

Figure 2: The standard toric diagram for the generalized conifold $xy = uv = z^2$ (far left). To the right are six SL(3; $\mathbb{Z}$) transformations (a)–(f) thereof (drawn in red) and hence are equivalent toric diagrams for the variety. We embed these six diagrams into the abelian orbifold $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ in order to perform partial resolution and thus the gauge theory data.

2.2 Partial resolution

Let us see whether we can reproduce these field theories with the Inverse Algorithm. The toric diagram for $xy = z^2w^2$ is given in the very left of figure 2. Of course, the hypersurface is three complex-dimensional so there is actually an undrawn apex for the toric diagram, and each of the nodes is in fact a three-vector in $\mathbb{Z}^3$. Indeed the fact that it is locally Calabi-Yau that guarantees all the nodes to be coplanar. The next step is the realisation that it can be embedded into the well-known toric diagram for the abelian orbifold $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ consisting of 10 lattice points. The reader is referred to [6, 8] for the actual coördinates of the points, a detail which, though crucial, we shall not belabour here.

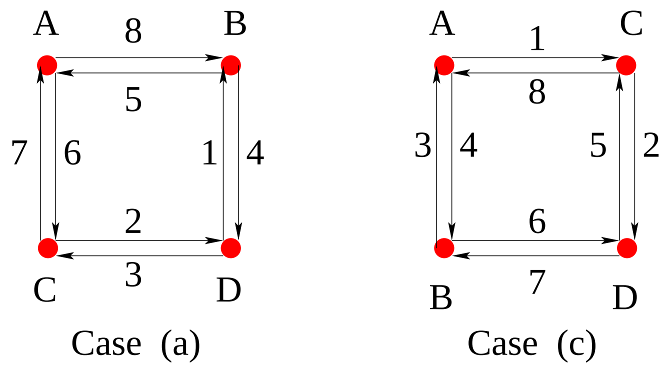

Figure 3: The quiver diagram encoding the matter content of Cases (a) and (c) of figure 2.

The important point is that there are six ways to embed our toric diagram into the orbifold one, all related by $\mathrm{SL}(3;\mathbb{Z})$ transformations. This is indicated in parts (a)–(f) of figure 2. We emphasise that these six diagrams, drawn in red, are *equivalent* descriptions of $xy = z^2w^2$ by virtue of their being unimodularly related; therefore they are all candidates for toric duality.

Now we use our Inverse Algorithm, by partially resolving $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$, to obtain the gauge theory data for the D-brane probing $xy = z^2w^2$. In summary, after exploring the six possible partial resolutions, we find that cases (a) and (b) give identical results, while $(c{,}d{,}e{,}f)$ give the same result which is inequivalent from $(a{,}b)$. Therefore we conclude that cases (a) and (c) are inequivalent torically dual theories for $xy = z^2w^2$. In the following we detail the data for these two contrasting cases. We refer the reader to [6, 8] for details and notation.

2.3 Case (a) from partial resolution

For case (a), the matter content is encoded the d-matrix which indicates the charges of the 8 bi-fundamentals under the 4 gauge groups. This is the incidence matrix for the quiver diagram drawn in part (a) of figure 3.

$$\begin{array}{c|cccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ \hline \mathrm{U}(1)_A & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ \mathrm{U}(1)_B & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ \mathrm{U}(1)_C & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ \mathrm{U}(1)_D & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

On the other hand, the F-terms are encoded in the K-matrix

$$\begin{array}{cccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}$$

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From K we get two relations $X_5X_8 = X_6X_7$ and $X_1X_4 = X_2X_3$ (these are the relations one must impose on the quiver to obtain the final variety; equivalently, they correspond to the F-term constraints arising from the superpotential). Notice that here each term is chargeless under all 4 gauge groups, so when we integrate back to get the superpotential, we should multiply by chargeless quantities also.[1]

The relations must come from the F-flatness $\frac{\partial}{\partial X_i}W = 0$ and thus we can use these relations to integrate back to the superpotential W. However we meet some ambiguities.[2] In principle we can have two different choices:

$$\begin{array}{ll} \text{(i)} & W_1 = (X_5X_8 - X_6X_7)(X_1X_4 - X_2X_3) \\ \text{(ii)} & W_2 = \psi_1(X_5X_8 - X_6X_7) + \psi_2(X_1X_4 - X_2X_3)\,, \end{array}$$

where for now ψ_i are simply chargeless fields.

We shall evoke physical arguments to determine which is correct. Expanding (i) gives $W_1 = X_5X_8X_1X_4 - X_6X_7X_1X_4 - X_5X_8X_2X_3 + X_6X_7X_2X_3$. Notice the term $X_6X_7X_1X_4$: there is no common gauge group under which these four fields are charged, i.e. these 4 arrows (q. v. figure 3) do not intersect at a single node. This makes (i) very unnatural and exclude it.

Case (ii) does not have the above problem and indeed all four fields X_5, X_8, X_6, X_7 are charged under the $\mathrm{U}(1)_A$ gauge group, so considering ψ_1 to be an adjoint of $\mathrm{U}(1)_A$, we do obtain a physically meaningful interaction. Similarly ψ_2 will be the adjoint of $\mathrm{U}(1)_D$, interacting with X_1, X_4, X_2, X_3.

However, we are not finish yet. From figure 3 we see that X_5, X_8, X_1, X_4 are all charged under $\mathrm{U}(1)_B$, while X_6, X_7, X_2, X_3 are all charged under $\mathrm{U}(1)_C$. From a physical point of view, there should be some interaction terms between these fields. Possibilities are $X_5X_8X_1X_4$ and $X_6X_7X_2X_3$. To add these terms into W_2 is very easy, we simply perform the following replacement:[3] $\psi_1 \longrightarrow \psi_1 - X_1X_4$, $\psi_2 \longrightarrow \psi_2 - X_6X_7$. Putting everything together, we finally obtain that Case (a) has matter content as described in figure 3 and the superpotential

$$W = (\psi_1 - X_1X_4)(X_5X_8 - X_6X_7) + (\psi_2 - X_6X_7)(X_1X_4 - X_2X_3)\,. \tag{2.3}$$

This is precisely the theory (a) from the brane setup in the last section! Comparing (2.3) with (2.1), we see that they are exact same under the following redefinition of variables:

$$\begin{array}{ccc} B_1, B_2 \Longleftrightarrow X_5, X_8 & C_1, C_2 \Longleftrightarrow X_6, X_7 & D_1, D_2 \Longleftrightarrow X_2, X_3 \\ A_1, A_2 \Longleftrightarrow X_1, X_4 & \phi_2 \Longleftrightarrow \psi_1 & \phi_1 \Longleftrightarrow \psi_2\,. \end{array}$$

[1] In more general situations the left- and right-hand sides may not be singlets, but transform in the same gauge representation.

[2] The ambiguities arise because in the abelian case (toric language) the adjoints are chargeless. In fact, no ambiguity arises if one performs the Higgsing associated to the partial resolution in the non-abelian case. We have performed this exercise in cases (a) and (c), and verified the result obtained by the different argument offered in the text.

[3] Here we choose the sign purposefully for later convenience. However, we do need, for the cancellation of the unnatural interaction term $X_1X_4X_6X_7$, that they both have the same sign.

In conclusion, case (a) of our Inverse Algorithm reproduces the results of case (a) of the brane setup.

2.4 Case (c) from partial resolution

For case (c), the matter content is given by the quiver in figure 3, which has the charge matrix d equal to

$$\left(\begin{array}{c|cccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ \mathrm{U}(1)_A & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ \mathrm{U}(1)_B & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ \mathrm{U}(1)_C & 1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \\ \mathrm{U}(1)_D & 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 \end{array}\right).$$

This is precisely the matter content of case (b) of the brane setup. The F-terms are given by

$$K = \left(\begin{array}{cccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right).$$

From it we can read out the relations $X_1X_8 = X_6X_7$ and $X_2X_5 = X_3X_4$. Again there are two ways to write down the superpotential

$$\begin{aligned} &\text{(i)} \quad W_1 = (X_1X_8 - X_6X_7)(X_3X_4 - X_2X_5) \\ &\text{(ii)} \quad W_2 = \psi_1(X_1X_8 - X_6X_7) + \psi_2(X_3X_4 - X_2X_5)\,. \end{aligned}$$

In this case, because X_1, X_8, X_6, X_7 are not charged under any common gauge group, it is impossible to include any adjoint field ψ to give a physically meaningful interaction and so (ii) is unnatural. We are left the superpotential W_1. Indeed, comparing with (2.2), we see they are identical under the redefinitions

$$\begin{aligned} A_1, A_2 &\Longleftrightarrow X_1, X_8 \qquad & B_1, B_2 &\Longleftrightarrow X_3, X_4 \\ C_1, C_2 &\Longleftrightarrow X_6, X_7 \qquad & D_1, D_2 &\Longleftrightarrow X_2, X_5 \end{aligned}$$

Therefore we have reproduced case (b) of the brane setup.

What have we achieved? We have shown that toric duality due to inequivalent embeddings of unimodularly related toric diagrams for the generalized conifold $xy = z^2w^2$ gives two inequivalent physical world-volume theories on the D-brane probe, exemplified by cases (a) and (c). On the other hand, there are two T-dual brane setups for this singularity, also giving two inequivalent field theories (a) and (b). Upon comparison, case (a) (resp. (c)) from the Inverse Algorithm beautifully corresponds to case (a) (resp. (b)) from the brane setup. Somehow, a seemingly harmless trick in mathematics relates inequivalent brane setups. In fact we can say much more.

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3. Seiberg duality versus toric duality

As follows from [13], the two theories from the brane setups are actually related by Seiberg Duality [16], as pointed out in [11] (see also [17, 18]). Let us first review the main features of this famous duality, for unitary gauge groups.

Seiberg duality is a non-trivial infrared equivalence of $\mathcal{N} = 1$ supersymmetric field theories, which are different in the ultraviolet, but flow the the same interacting fixed point in the infrared. In particular, the very low energy features of the different theories, like their moduli space, chiral ring, global symmetries, agree for Seiberg dual theories. Given that toric dual theories, by definition, have identical moduli spaces, etc., it is natural to propose a connection between both phenomena.

The prototypical example of Seiberg duality is $\mathcal{N} = 1$ $\mathrm{SU}(N_c)$ gauge theory with N_f vector-like fundamental flavours, and no superpotential. The global chiral symmetry is $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, so the matter content quantum numbers are

	$\mathrm{SU}(N_c)$	$\mathrm{SU}(N_f)_L$	$\mathrm{SU}(N_f)_R$
Q	$\square$	$\square$	1
Q'	$\overline{\square}$	1	$\overline{\square}$

In the conformal window, $3N_c/2 \leq N_f \leq 3N_c$, the theory flows to an interacting infrared fixed point. The dual theory, flowing to the same fixed point is given $N = 1$ $\mathrm{SU}(N_f - N_c)$ gauge theory with N_f fundamental flavours, namely

	$\mathrm{SU}(N_f - N_c)$	$\mathrm{SU}(N_f)_L$	$\mathrm{SU}(N_f)_R$
q	$\square$	$\overline{\square}$	1
q'	$\overline{\square}$	1	$\square$
M	1	$\square$	$\overline{\square}$

and superpotential $W = Mqq'$. From the matching of chiral rings, the 'mesons' M can be thought of as composites QQ' of the original quarks.

It is well established [13], that in an $\mathcal{N} = 1$ (IIA) brane setup for the four dimensional theory such as figure 1, Seiberg duality is realised as the crossing of 2 non-parallel NS-NS$'$ branes. In other words, as pointed out in [11], cases (a) and (b) are in fact a Seiberg dual pair. Therefore it seems that the results from the previous section suggest that toric duality is a guise of Seiberg duality, for theories with moduli space admitting a toric descriptions. It is therefore the intent of the remainder of this paper to examine and support

Conjecture. *Toric duality is Seiberg duality for $\mathcal{N} = 1$ theories with toric moduli spaces.*

4. Partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and Seiberg duality

Let us proceed to check more examples. So far the other known examples of torically dual theories are from various partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. In particular it was found in [8] that the (complex) cones over the zeroth Hirzebruch surface as well as the second del Pezzo surface each has two toric dual pairs. We remind the reader of these theories.

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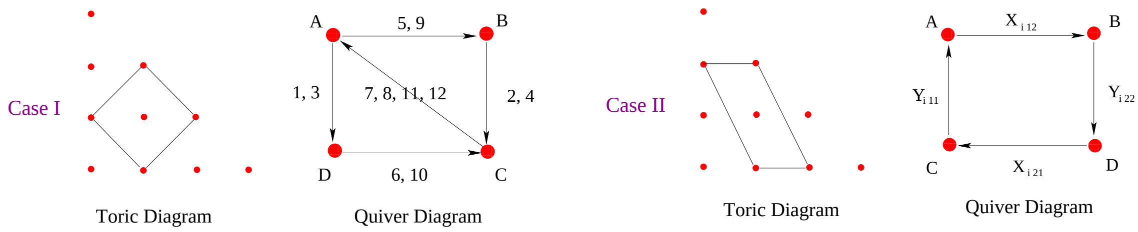

Figure 4: The quiver and toric diagrams of the 2 torically dual theories corresponding to the cone over the zeroth Hirzebruch surface F_0.

	Matter content d	Superpotential
I	$\begin{array}{c\|cccccccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} \\ \hline A & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & 1 \\ B & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ D & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}$	$X_1X_8X_{10} - X_3X_7X_{10} - X_2X_8X_9 - X_1X_6X_{12} + X_3X_6X_{11} + X_4X_7X_9 + X_2X_5X_{12} - X_4X_5X_{11}$
II	$\begin{array}{c\|cccccccc} & X_{112} & Y_{122} & Y_{222} & Y_{111} & Y_{211} & X_{121} & X_{212} & X_{221} \\ \hline A & -1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ B & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 \\ D & 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 \end{array}$	$\epsilon^{ij}\epsilon^{kl} X_{i\ 12} Y_{k\ 22} X_{j\ 21} Y_{l\ 11}$

Table 1: The matter content and interaction for the ywo phases of the Zeroth Hirzebruch surface.

4.1 Hirzebruch zero

There are two torically dual theories for the cone over the zeroth Hirzebruch surface F_0. The toric and quiver diagrams are given in figure 4, the matter content and interactions are given in table 1

Let us use the field theory rules from section 3 on Seiberg Duality to examine these two cases in detail. The charges of the matter content for case II, upon promotion from U(1) to SU(N)[4] (for instance, following the partial resolution in the non-abelian case, as in [3, 4]), can be re-written as (redefining fields $(X_i, Y_i, Z_i, W_i) := (X_{i\ 12}, Y_{i\ 22}, X_{i\ 21}, Y_{i\ 11})$ with $i = 1, 2$ and gauge groups $(a, b, c, d) := (A, C, B, D)$ for convenience):

	SU$(N)_a$	SU$(N)_b$	SU$(N)_c$	SU$(N)_d$
X_i	$\square$	$\overline{\square}$		
Y_i		$\square$	$\overline{\square}$	
Z_i			$\square$	$\overline{\square}$
W_i	$\overline{\square}$			$\square$

The superpotential is then

$$W_{\rm II} = X_1Y_1Z_2W_2 - X_1Y_2Z_2W_1 - X_2Y_1Z_1W_2 + X_2Y_2Z_1W_1 \,.$$

Let us dualise with respect to the a gauge group. This is a SU(N) theory with $N_c = N$ and $N_f = 2N$ (as there are two X_i's). The chiral symmetry is however broken from

[4] Concerning the U(1) factors, these are in fact generically absent, since they are anomalous in the original $\mathbb{Z}_3 \times \mathbb{Z}_3$ singularity, and the Green-Schwarz mechanism canceling their anomaly makes them massive [19] (see [20, 21, 22] for an analogous 6d phenomenon). However, there is a well-defined sense in which one can use the abelian case to study the toric moduli space [3].

$\mathrm{SU}(2N)_L \times \mathrm{SU}(2N)_R$ to $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$, which moreover is gauged as $\mathrm{SU}(N)_b \times \mathrm{SU}(N)_d$. Ignoring the superpotential W_{II}, the dual theory would be:

$$
\begin{array}{c|cccc}
 & \mathrm{SU}(N)_{a'} & \mathrm{SU}(N)_b & \mathrm{SU}(N)_c & \mathrm{SU}(N)_d \\
\hline
q_i & \overline{\square} & \square & & \\
Y_i & & \square & \overline{\square} & \\
Z_i & & & \square & \overline{\square} \\
q'_i & \square & & & \overline{\square} \\
M_{ij} & & \overline{\square} & & \square
\end{array}
\tag{4.1}
$$

We note that there are M_{ij} giving 4 bi-fundamentals for bd. They arise from the Seiberg mesons in the bi-fundamental of the enhanced chiral symmetry $\mathrm{SU}(2N) \times \mathrm{SU}(2N)$, once decomposed with respect to the unbroken chiral symmetry group. The superpotential is

$$W' = M_{11} q_1 q'_1 - M_{12} q_2 q'_1 - M_{21} q_1 q'_2 + M_{22} q_2 q'_2 \,.$$

The choice of signs in W' will be explained shortly.

Of course, W_{II} is not zero and so give rise to a deformation in the original theory, analogous to those studied in e.g. [15]. In the dual theory, this deformation simply corresponds to W_{II} rewritten in terms of mesons, which can be thought of as composites of the original quarks, i.e., $M_{ij} = W_i X_j$. Therefore we have

$$W_{\mathrm{II}} = M_{21} Y_1 Z_2 - M_{11} Y_2 Z_2 - M_{22} Y_1 Z_1 + M_{12} Y_2 Z_1$$

which is written in the new variables. The rule for the signs is that e.g. the field M_{21} appears with positive sign in W_{II}, hence it should appear with negative sign in W', and analogously for others. Putting them together we get the superpotential of the dual theory

$$
\begin{aligned}
W_{\mathrm{II}}^{\mathrm{dual}} &= W_{\mathrm{II}} + W' \\
&= M_{11} q_1 q'_1 - M_{12} q_2 q'_1 - M_{21} q_1 q'_2 + M_{22} q_2 q'_2 + M_{21} Y_1 Z_2 - \\
&\quad - M_{11} Y_2 Z_2 - M_{22} Y_1 Z_1 + M_{12} Y_2 Z_1
\end{aligned}
\tag{4.2}
$$

Upon the field redefinitions

$$
\begin{array}{rlrlrlrl}
M_{11} & \to X_7 & M_{12} & \to X_8 & M_{21} & \to X_{11} & M_{22} & \to X_{12} \\
q_1 & \to X_4 & q_2 & \to X_2 & q_{1'} & \to X_9 & q_{2'} & \to X_5 \\
Y_1 & \to X_6 & Y_2 & \to X_{10} & Z_1 & \to X_1 & Z_2 & \to X_3
\end{array}
$$

we have the field content (4.1) and superpotential (4.2) matching precisely with case I in table 1. We conclude therefore that the two torically dual cases I and II obtained from partial resolutions are indeed Seiberg duals!

4.2 del Pezzo 2

Encouraged by the results above, let us proceed with the cone over the second del Pezzo surface, which also have 2 torically dual theories. The toric and quiver diagrams are given in figure 5.

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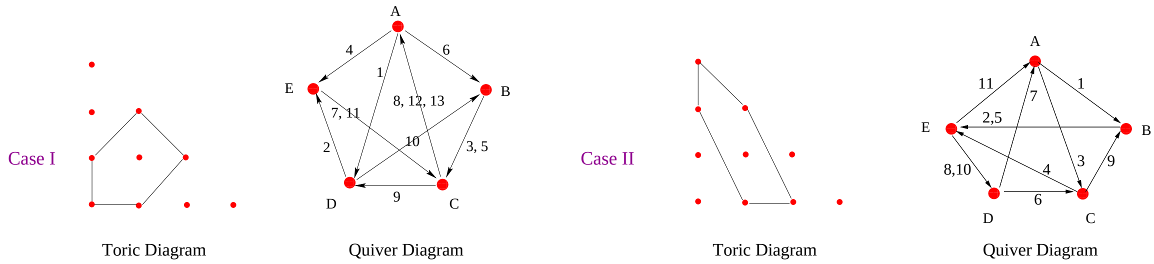

Figure 5: The quiver and toric diagrams of the 2 torically dual theories corresponding to the cone over the second del Pezzo surface.

Again we start with Case II. Working analogously, upon dualisation on node D neglecting the superpotential, the matter content of II undergoes the following change:

	$SU(N)_A$	$SU(N)_B$	$SU(N)_C$	$SU(N)_D$	$SU(N)_E$
X_1	$\overline{\square}$	$\square$			
X_2		$\overline{\square}$			$\square$
X_5		$\overline{\square}$			$\square$
X_3	$\overline{\square}$		$\square$		
X_4			$\overline{\square}$		$\square$
X_9		$\square$	$\overline{\square}$		
X_{11}	$\square$				$\overline{\square}$
X_6			$\square$	$\overline{\square}$	
X_7	$\square$			$\overline{\square}$	
X_8				$\square$	$\overline{\square}$
X_{10}				$\square$	$\overline{\square}$

dual on D $\Longrightarrow$

	$SU(N)_A$	$SU(N)_B$	$SU(N)_C$	$SU(N)_D$	$SU(N)_E$
X_6	$\overline{\square}$	$\square$			
X_5		$\overline{\square}$			$\square$
X_3		$\overline{\square}$			$\square$
X_1	$\overline{\square}$		$\square$		
X_4			$\overline{\square}$		$\square$
X_{10}		$\square$	$\overline{\square}$		
X_{13}	$\square$				$\overline{\square}$
$\widetilde{X}_6$			$\overline{\square}$	$\square$	
$\widetilde{X}_7$	$\overline{\square}$			$\square$	
$\widetilde{X}_8$				$\overline{\square}$	$\square$
$\widetilde{X}_{10}$				$\overline{\square}$	$\square$
$M_{EA,1}$	$\square$				$\overline{\square}$
$M_{EA,2}$	$\square$				$\overline{\square}$
$M_{EC,1}$			$\square$		$\overline{\square}$
$M_{EC,2}$			$\square$		$\overline{\square}$

(4.3)

Let us explain the notations in (4.3). We use the field labelling of I for the left and that of II for after the dualisation. Before Seiberg duality we have 11 fields $X_{1,\ldots,11}$. After the dualisation on gauge group D, the we obtain dual quarks (corresponding to bi-fundamentals conjugate to the original quark X_6, X_7, X_8, X_{10}) which we denote $\widetilde{X}_6, \widetilde{X}_7, \widetilde{X}_8, \widetilde{X}_{10}$. Further-

<table>
<tr><th></th><th>Matter Content d</th><th>Superpotential</th></tr>
<tr><td>I</td><td>$\begin{array}{c|ccccccccccccc} & Y_1 & Y_2 & Y_3 & Y_4 & Y_5 & Y_6 & Y_7 & Y_8 & Y_9 & Y_{10} & Y_{11} & Y_{12} & Y_{13} \\ \hline A & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ B & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ D & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ E & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}$</td><td>$Y_2Y_9Y_{11} - Y_9Y_3Y_{10} - Y_4Y_8Y_{11} - Y_1Y_2Y_7Y_{13}+$
$+Y_{13}Y_3Y_6 - Y_5Y_{12}Y_6 + Y_1Y_5Y_8Y_{10} + Y_4Y_7Y_{12}$</td></tr>
<tr><td>II</td><td>$\begin{array}{c|ccccccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} \\ \hline A & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ B & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ C & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ D & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & 0 \\ E & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{array}$</td><td>$X_5X_8X_6X_9 + X_1X_2X_{10}X_7 + X_{11}X_3X_4-$
$-X_4X_{10}X_6 - X_2X_8X_7X_3X_9 - X_{11}X_1X_5$</td></tr>
</table>

Table 2: The matter content and interaction for the ywo phases of the Second del Pezzo Surface.

more we have added meson fields $M_{EA,1}, M_{EA,2}, M_{EC,1}, M_{EC,2}$, which are Seiberg mesons decomposed with respect to the unbroken chiral symmetry group.

As before, one should incorporate the interactions as a deformation of this duality. Naïvely we have 15 fields in the dual theory, but as we will show below, the resulting superpotential provides a mass term for the fields X_4 and $M_{EC,2}$, which transform in conjugate representations. Integrating them out, we will be left with 13 fields, the number of fields in Case I. In fact, with the mapping

dual of II	X_1	X_2	X_5	X_3	X_4	X_9	X_{11}	$\widetilde{X}_6$	$\widetilde{X}_7$	$\widetilde{X}_8$	$\widetilde{X}_{10}$
Case I	Y_6	Y_5	Y_3	Y_1	massive	Y_{10}	Y_{13}	Y_2	Y_4	Y_{11}	Y_7

and

dual of II	$M_{EA,1}$	$M_{EA,2}$	$M_{EC,1}$	$M_{EC,2}$
Case I	Y_8	Y_{12}	Y_9	massive

we conclude that the matter content of the Case II dualised on gauge group D is identical to Case I!

Let us finally check the superpotentials, and also verify the claim that X_4 and $M_{EC,2}$ become massive. Rewriting the superpotential of II from (2) in terms of the dual variables (matching the mesons as composites $M_{EA,1} = X_8X_7$, $M_{EA,2} = X_{10}X_7$, $M_{EC,1} = X_8X_6$, $M_{EC,2} = X_{10}X_6$), we have

$$\begin{aligned} W_{\rm II} &= X_5M_{EC,1}X_9 + X_1X_2M_{EA,2} + X_{11}X_3X_4 - \\ &\quad - X_4M_{EC,2} - X_2M_{EA,1}X_3X_9 - X_{11}X_1X_5 \,. \end{aligned}$$

As is with the previous subsection, to the above we must add the meson interaction terms coming from Seiberg duality, namely

$$W_{\rm meson} = M_{EA,1}\widetilde{X}_7\widetilde{X}_8 - M_{EA,2}\widetilde{X}_7\widetilde{X}_{10} - M_{EC,1}\widetilde{X}_6\widetilde{X}_8 + M_{EC,2}\widetilde{X}_6\widetilde{X}_{10} \,,$$

(notice again the choice of sign in $W_{\rm meson}$). Adding this two together we have

$$\begin{aligned} W_{\rm II}^{\rm dual} &= X_5M_{EC,1}X_9 + X_1X_2M_{EA,2} + X_{11}X_3X_4 - \\ &\quad - X_4M_{EC,2} - X_2M_{EA,1}X_3X_9 - X_{11}X_1X_5 + \\ &\quad + M_{EA,1}\widetilde{X}_7\widetilde{X}_8 - M_{EA,2}\widetilde{X}_7\widetilde{X}_{10} - M_{EC,1}\widetilde{X}_6\widetilde{X}_8 + M_{EC,2}\widetilde{X}_6\widetilde{X}_{10} \,. \end{aligned}$$

Now it is very clear that both X_4 and $M_{EC,2}$ are massive and should be integrated out:

$$X_4 = \widetilde{X}_6 \widetilde{X}_{10}\,, \qquad M_{EC,2} = X_{11} X_3\,.$$

Upon substitution we finally have

$$\begin{aligned} W_{\rm II}^{\rm dual} = &\ X_5 M_{EC,1} X_9 + X_1 X_2 M_{EA,2} + X_{11} X_3 \widetilde{X}_6 \widetilde{X}_{10} - X_2 M_{EA,1} X_3 X_9 - \\ &- X_{11} X_1 X_5 + M_{EA,1} \widetilde{X}_7 \widetilde{X}_8 - M_{EA,2} \widetilde{X}_7 \widetilde{X}_{10} - M_{EC,1} \widetilde{X}_6 \widetilde{X}_8\,, \end{aligned}$$

which with the replacement rules given above we obtain

$$\begin{aligned} W_{\rm II}^{\rm dual} = &\ Y_3 Y_9 Y_{10} + Y_6 Y_5 Y_{12} + Y_{13} Y_1 Y_2 Y_7 - Y_5 Y_1 Y_{10} Y_8 - \\ &- Y_{13} Y_6 Y_3 + Y_8 Y_4 Y_{11} - Y_{12} Y_4 Y_7 - Y_9 Y_2 Y_{11}\,. \end{aligned}$$

This we instantly recognise, by referring to (2), as the superpotential of Case I.

In conclusion therefore, with the matching of matter content and superpotential, the two torically dual cases I and II of the cone over the second del Pezzo surface are also Seiberg duals.

5. Brane diamonds and Seiberg duality

Having seen the above arguments from field theory, let us support that toric duality is Seiberg duality from yet another perspective, namely, through brane setups. The use of this T-dual picture for D3-branes at singularities will turn out to be quite helpful in showing that toric duality reproduces Seiberg duality.

What we have learnt from the examples where a brane interval picture is available (i.e. NS- and D4-branes in the manner of [14]) is that the standard Seiberg duality by brane crossing reproduces the different gauge theories obtained from toric arguments (different partial resolutions of a given singularity). Notice that the brane crossing corresponds, under T-duality, to a change of the B field in the singularity picture, rather than a change in the singularity geometry [11, 17]. Hence, the two theories arise on the world-volume of D-branes probing the same singularity.

Unfortunately, brane intervals are rather limited, in that they can be used to study Seiberg duality for generalized conifold singularities, $xy = z^k w^l$. Although this is a large class of models, not many examples arise in the partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. Hence the relation to toric duality from partial resolutions cannot be checked for most examples.

Therefore it would be useful to find other singularities for which a nice T-dual brane picture is available. Nice in the sense that there is a motivated proposal to realize Seiberg duality in the corresponding brane setup. A good candidate for such a brane setup is *brane diamonds*, studied in [23].

Reference [24] (see also [25, 26]) introduced brane box configurations of intersecting NS- and NS'-branes (spanning 012345 and 012367, respectively), with D5-branes (spanning 012346) suspended among them. Brane diamonds [23] generalized (and refined) this setup by considering situations where the NS- and the NS'-branes recombine and span a smooth

Figure 6: Seiberg duality from the brane diamond construction for the generalized conifold $xy = z^2w^2$. Part (I) corresponds to the brane interval picture with alternating ordering of NS- and NS′-branes, whereas part (II) matches the other ordering.

holomorphic curve in the 4567 directions, in whose holes D5-branes can be suspended as soap bubbles. Typical brane diamond pictures are as in figures in the remainder of the paper.

Brane diamonds are related by T-duality along 46 to a large set of D-branes at singularities. With the set of rules to read off the matter content and interactions in [23], they provide a useful pictorial representation of these D-brane gauge field theories. In particular, they correspond to singularities obtained as the abelian orbifolds of the conifold studied in [11, section 5], and partial resolutions thereof. Concerning this last point, brane diamond configurations admit two kinds of deformations: motions of diamond walls in the directions 57, and motions of diamond walls in the directions 46. The former T-dualize to geometric sizes of the collapse cycles, hence trigger partial resolutions of the singularity (notice that when a diamond wall moves in 57, the suspended D5-branes snap back and two gauge factors recombine, leading to a Higgs mechanism, triggered by FI terms). The later do not modify the T-dual singularity geometry, and correspond to changes in the B-fields in the collapsed cycles.

The last statement motivates the proposal made in [23] for Seiberg duality in this setup. It corresponds to closing a diamond, while keeping it in the 46 plane, and reopening it with the opposite orientation. The orientation of a diamond determines the chiral multiplets and interactions arising from the picture. The effect of this is shown in fig 7 of [23]: The rules are

1. When the orientation of a diamond is flipped, the arrows going in or out of it change orientation;

2. one has to include/remove additional arrows to ensure a good 'arrow flow' (ultimately connected to anomalies, and to Seiberg mesons)

3. Interactions correspond to closed loops of arrows in the brane diamond picture.

4. In addition to these rules, and based in our experience with Seiberg duality, we propose that when in the final picture some mesons appear in gauge representations conjugate to some of the original field, the conjugate pair gets massive.

These rules reproduce Seiberg duality by brane crossing in cases where a brane interval picture exists. In fact, one can reproduce our previous discussion of the $xy = z^2w^2$ in this

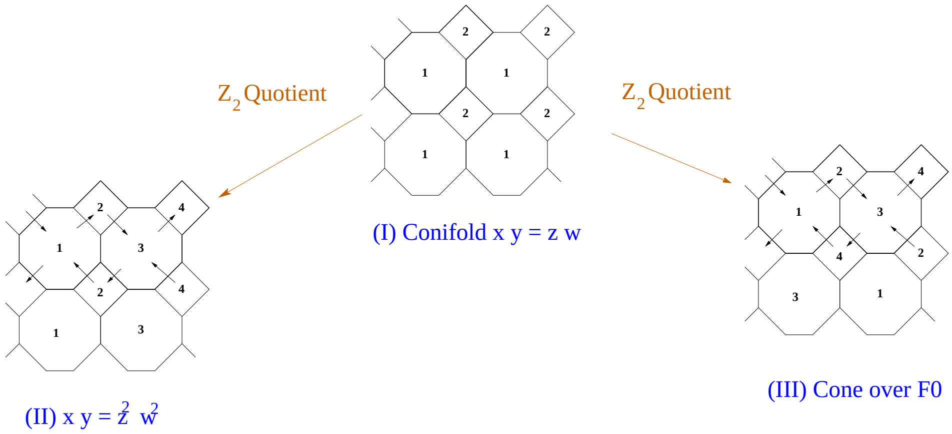

Figure 7: (I) Brane diamond for the conifold. Identifications in the infinite periodic array of boxes leads to a two-diamond unit cell, whose sides are identified in the obvious manner. From (I) we have 2 types of $\mathbb{Z}_2$ quotients: (II) Brane diamond for the $\mathbb{Z}_2$ quotient of the conifold $xy = z^2w^2$, which is a case of the so-called generalised conifold. The identifications of sides are trivial, not tilting. The final spectrum is the familiar non-chiral spectrum for a brane interval with two NS and two NS' branes (in the alternate configuration); (III) Brane diamond for the $\mathbb{Z}_2$ quotient of the conifold yielding the complex cone over F_0. The identifications of sides are shifted, a fact related to the specific 'tilted' refinement of the toric lattice.

language, as shown in figure 6. Notice that in analogy with the brane interval case the diamond transition proposed to reproduce Seiberg duality does not involve changes in the T-dual singularity geometry, hence ensuring that the two gauge theories will have the same moduli space.

Let us re-examine our aforementioned examples.

5.1 Brane diamonds for D3-branes at the cone over F_0

Now let us show that diamond Seiberg duality indeed relates the two gauge theories arising on D3-branes at the singularity which is a complex cone over F_0. The toric diagram of F_0 is similar to that of the conifold, only that it has an additional point (ray) in the middle of the square. Hence, it can be obtained from the conifold diagram by simply refining the lattice (by a vector $(1/2, 1/2)$ if the conifold lattice is generated by $(1,0)$, $(0,1)$). This implies [27]) that the space can be obtained as a $\mathbb{Z}_2$ quotient of the conifold, specifically modding $xy = zw$ by the action that flips all coordinates.

Performing two T-dualities in the conifold one reaches the brane diamond picture described in [23, figure 5], which is composed by two-diamond cell with sides identified, see part (I) of figure 7.

However, we are interested not in the conifold but on a $\mathbb{Z}_2$ quotient thereof. Quotienting a singularity amounts to including more diamonds in the unit cell, i.e. picking a larger unit cell in the periodic array. There are two possible ways to do so, corresponding to two different $\mathbb{Z}_2$ quotients of the conifold. One corresponds to the generalized conifold $xy = z^2w^2$ encountered above, and whose diamond picture is given in part (II) of figure 7 for

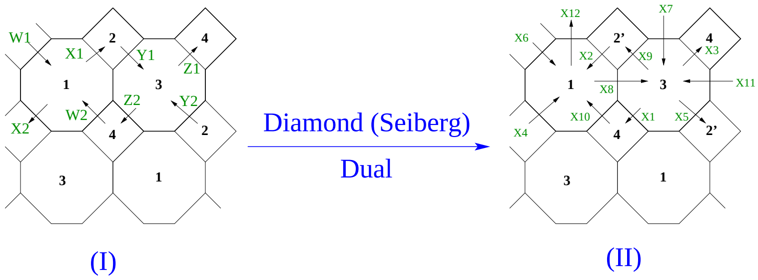

Figure 8: Brane diamond for the two cases of the cone over F_0. (I) is as in figure 7 and (II) is the result after the diamond duality. The resulting spectrum and interactions are those of the toric dual (and also Seiberg dual) of the initial theory (I).

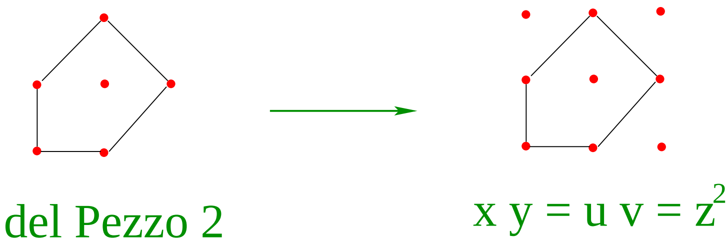

Figure 9: Embedding the toric diagram of dP2 into the orbifolded conifold described as $xy = v^2$, $uv = z^2$.

completeness. The second possibility is shown in part (III) of figure 7 and does correspond to the T-dual of the complex cone over F_0, so we shall henceforth concentrate on this case. Notice that the identifications of sides of the unit cell are shifted. The final spectrum agrees with the quiver before eq. (2.2) in [6]. Moreover, following [23], these fields have quartic interactions, associated to squares in the diamond picture, with signs given by the orientation of the arrow flow. They match the ones in case II in table 1.

Now let us perform the diamond duality in the box labeled 2. Following the diamond duality rules above, we obtain the result shown in figure 8. Careful comparison with the spectrum and interactions of case I in table 1, and also with the Seiberg dual computed in section 4.1 shows that the new diamond picture reproduces the toric dual/Seiberg dual of the initial one. Hence, brane diamond configurations provide a new geometric picture for this duality.

5.2 Brane diamonds for D3-branes at the cone over dP_2

The toric diagram for dP_2 shows it cannot be constructed as a quotient of the conifold. However, it is a partial resolution of the orbifolded conifold described as $xy = v^2$, $uv = z^2$ in $\mathbb{C}^5$ (we refer the reader to figure 9. This is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotient of the conifold whose brane diamond, shown in part (I) of figure 10, contains 8 diamonds in its unit cell.

Partial resolutions in the brane diamond language correspond to partial Higgsing, namely recombination of certain diamonds. As usual, the difficult part is to identify which

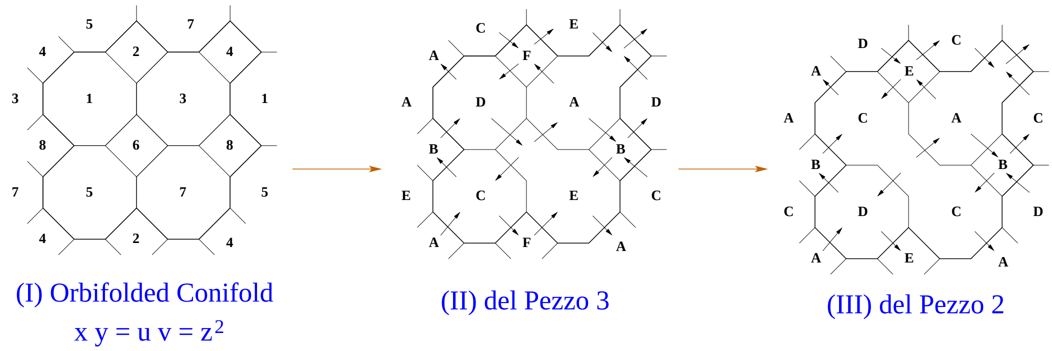

Figure 10: (I) Brane diamond for a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the conifold, namely $xy = z^2; uv = z^2$. From this we can partial resolve to (II) the cone over $dP3$ and thenceforth again to (III) the cone over dP_2, which we shall discuss in the context of Seiberg duality.

diamond recombination corresponds to which partial resolution. A systematic way proceed would be:[5]

1. Pick a diamond recombination;
2. Compute the final gauge theory;
3. Compute its moduli space, which should be the partially resolved singularity.

However, instead of being systematic, we prefer a shortcut and simply match the spectrum of recombined diamond pictures with known results of partial resolutions. In order to check we pick the right resolutions, it is useful to discuss the brane diamond picture for some intermediate step in the resolution to dP_2. A good intermediate point, for which the field theory spectrum is known is the complex cone over dP_3.

By trial and error matching, the diamond recombination which reproduces the world-volume spectrum for D3-branes at the cone over dP_3 (see [6, 8]), is shown in part (II) of figure 10. Performing a further resolution, chosen so as to match known results, one reaches the brane diamond picture for D3-branes on the cone over dP_2, shown in part (III) of figure 10. More specifically, the spectrum and interactions in the brane diamond configuration agrees with those of case I in (2).

This brane box diamond, obtained in a somewhat roundabout way, is our starting point to discuss possible dual realizations. In fact, recall that there is a toric dual field theory for dP_2, given as case II in (2). After some inspection, the desired effect is obtained by applying diamond Seiberg duality to the diamond labeled B. The corresponding process and the resulting diamond picture are shown in figure 11. Two comments are in order: notice that in applying diamond duality using the rules above, some vector-like pairs of fields have to be removed from the final picture; in fact one can check by field theory

[5]As an aside, let us remark that the use of brane diamonds to follow partial resolutions of singularities may provide an alternative to the standard method of partial resolutions of orbifold singularities [3, 6]. The existence of a brane picture for partial resolutions of orbifolded conifolds may turn out to be a useful advantage in this respect.

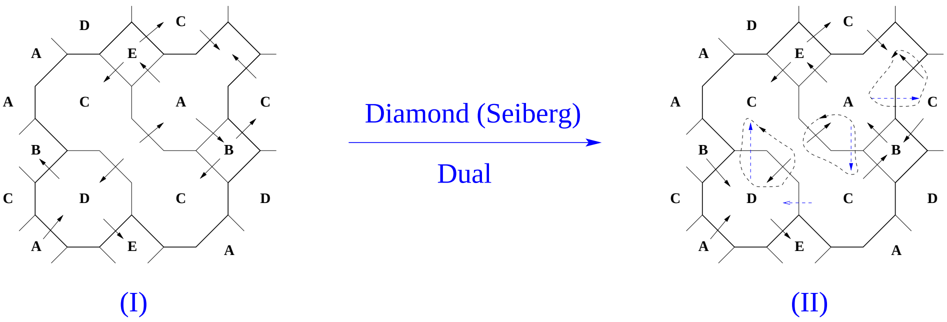

Figure 11: The brane diamond setup for the Seiberg dual configurations of the cone over dP_2. (I) is as in figure 10 and (II) is the results after Seiberg (diamond) duality and gives the spectrum for the toric dual theory. The added meson fields are drawn in dashed blue lines. Notice that applying the diamond dual rules carelessly one gets some additional vectorlike pairs, shown in the picture within dotted lines. Such multiplets presumably get massive in the Seiberg dualization, hence we do not consider them in the quiver.

Seiberg duality that the superpotential makes them massive. Second, notice that in this case we are applying duality in the direction opposite to that followed in the field theory analysis in section 4.2; it is not difficult to check that the field theory analysis works in this direction as well, namely the dual of the dual is the original theory.

Therefore this new example provides again a geometrical realization of Seiberg duality, and allows to connect it with Toric Duality.

We conclude this section with some remarks. The brane diamond picture presumably provides other Seiberg dual pairs by picking different gauge factors. All such models should have the same singularities as moduli space, and should be toric duals in a broad sense, even though all such toric duals may not be obtainable by partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. From this viewpoint we learn that Seiberg duality can provide us with new field theories and toric duals beyond the reach of present computational tools. This is further explored in section 7.

A second comment along the same lines is that Seiberg duality on nodes for which $N_f \neq 2N_c$ will lead to dual theories where some gauge factors have different rank. Taking the theory back to the 'abelian' case, some gauge factors turn out to be non-abelian. Hence, in these cases, even though Seiberg duality ensures the final theory has the same singularity as moduli space, the computation of the corresponding symplectic quotient is beyond the standard tools of toric geometry. Therefore, Seiberg duality can provide ('non-toric') gauge theories with toric moduli space.

6. A quiver duality from Seiberg duality

If we are not too concerned with the superpotential, when we make the Seiberg duality transformation, we can obtain the matter content very easily at the level of the quiver diagram. What we obtain are rules for a so-called "quiver duality" which is a rephrasing of the Seiberg duality transformations in field (brane diamond) theory in the language of

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quivers. Denote $(N_c)_i$ the number of colors at the i^{th} node, and a_{ij} the number of arrows from the node i to the j (the adjacency matrix) The rules on the quiver to obtain Seiberg dual theories are

1. Pick the dualisation node i_0. Define the following sets of nodes: $I_{\rm in}$:= nodes having arrows going into i_0; $I_{\rm out}$:= those having arrow coming from i_0 and I_{no} := those unconnected with i_0. The node i_0 should not be included in this classification.

2. Change the rank of the node i_0 from N_c to $N_f - N_c$ where N_f is the number of vector-like flavours, $N_f = \sum_{i\in I_{\rm in}} a_{i,i_0} = \sum_{i\in I_{\rm out}} a_{i_0,i}$

3. Reverse all arrows going in or out of i_0, therefore
$$a_{ij}^{\rm dual} = a_{ji} \qquad \text{if either } i,j = i_0$$

4. Only arrows linking $I_{\rm in}$ to $I_{\rm out}$ will be changed and all others remain unaffected.

5. For every pair of nodes A, B, $A \in I_{\rm out}$ and $B \in I_{\rm in}$, change the number of arrows a_{AB} to
$$a_{AB}^{\rm dual} = a_{AB} - a_{i_0 A} a_{B i_0} \qquad \text{for } A \in I_{\rm out}\,, \quad B \in I_{\rm in}\,.$$
If this quantity is negative, we simply take it to mean $-a^{\rm dual}$ arrow go from B to A.

These rules follow from applying Seiberg duality at the field theory level, and therefore are consistent with anomaly cancellation. In particular, notice that for any node $i \in I_{\rm in}$, we have replaced $a_{i,i_0} N_c$ fundamental chiral multiplets by $-a_{i,i_0}(N_f - N_c) + \sum_{j\in I_{\rm out}} a_{i,i0} a_{i_0,j}$ which equals $-a_{i,i_0}(N_f - N_c) + a_{i,i0} N_f = a_{i,i_0} N_c$, and ensures anomaly cancellation in the final theory. Similarly for nodes $j \in I_{\rm out}$.

It is straightforward to apply these rules to the quivers in the by now familiar examples in previous sections.

In general, we can choose an arbitrary node to perform the above Seiberg duality rules. However, not every node is suitable for a toric description. The reason is that, if we start from a quiver whose every node has the same rank N, after the transformation it is possible that this no longer holds. We of course wish so because due to the very definition of the $\mathbb{C}^*$ action for toric varieties, toric descriptions are possible iff all nodes are U(1), or in the non-abelian version, SU(N). If for instance we choose to Seiberg dualize a node with $3N$ flavours, the dual node will have rank $3N - N = 2N$ while the others will remain with rank N, and our description would no longer be toric. For this reason we must choose nodes with only $2N_f$ flavors, if we are to remain within toric descriptions.

One natural question arises: if we Seiberg-dualise every possible allowed node, how many different theories will we get? Moreover how many of these are torically dual? Let we re-analyse the examples we have thus far encountered.

6.1 Hirzebruch zero

Starting from case (II) of F_0 (recall table 1) all of four nodes are qualified to yield toric Seiberg duals (they each have 2 incoming and 2 outgoing arrows and hence $N_f = 2N$).

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Dualising any one will give to case (I) of F_0. On the other hand, from (I) of F_0, we see that only nodes B, D are qualified to be dualized. Choosing either, we get back to the case (II) of F_0. In another word, cases (I) and (II) are closed under the Seiberg-duality transformation. In fact, this is a very strong evidence that there are only two toric phases for F_0 no matter how we embed the diagram into higher $\mathbb{Z}_k \times \mathbb{Z}_k$ singularities. This also solves the old question [6, 8] that the Inverse Algorithm does not in principle tell us how many phases we could have. Now by the closeness of Seiberg-duality transformations, we do have a way to calculate the number of possible phases. Notice, on the other hand, the existence of non-toric phases.

6.2 del Pezzo 0,1,2

Continuing our above calculation to del Pezzo singularities, we see that for dP_0 no node is qualified, so there is only one toric phase which is consistent with the standard result [8] as a resolution $\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{C}^3/\mathbb{Z}_3$. For dP_1, nodes A, B are qualified (all notations coming from [8]), but the dualization gives back to same theory, so it too has only one phase.

For our example dP_2 studied earlier (recall figure 2), there are four points A, B, C, D which are qualified in case (II). Nodes A, C give back to case (II) while nodes B, D give rise to case (I) of dP_2. On the other hand, for case (I), three nodes B, D, E are qualified. Here nodes B, E give case (II) while node D give case (I). In other words, cases (I) and (II) are also closed under the Seiberg-duality transformation, so we conclude that there too are only two phases for dP_2, as presented earlier.

6.3 The four phases of dP_3

Things become more complex when we discuss the phases of dP_3. As we remarked before, due to the running-time limitations of the Inverse Algorithm, only one phase was obtained in [8]. However, one may expect this case to have more than just one phase, and in fact a recent paper has given another phase [28]. Here, using the closeness argument we give evidence that there are four (toric) phases for dP_3. We will give only one phase in detail. Others are similarly obtained.

Starting from case (I) given in [8] and dualizing node B, (we refer the reader to figure 12) we get the charge (incidence) matrix d as

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}
\left(\begin{array}{cccccccccccccccccc}
q_1 & q_2 & q'_1 & q'_2 & X_1 & X_2 & X_7 & X_9 & X_{10} & X_{11} & M_1 & X_{14} & M_2 & X_8 & M'_1 & X_5 & X_{12} & M'_2 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

where

$$M_1 = X_4 X_3\,, \qquad M_2 = X_4 X_6\,, \qquad M'_1 = X_{13} X_3\,, \qquad M'_2 = X_{13} X_6$$

are the added mesons. Notice that X_{14} and M_2 have opposite charge. In fact, both are massive and will be integrate out. Same for pairs (X_8, M'_1) and (X_5, M'_2).

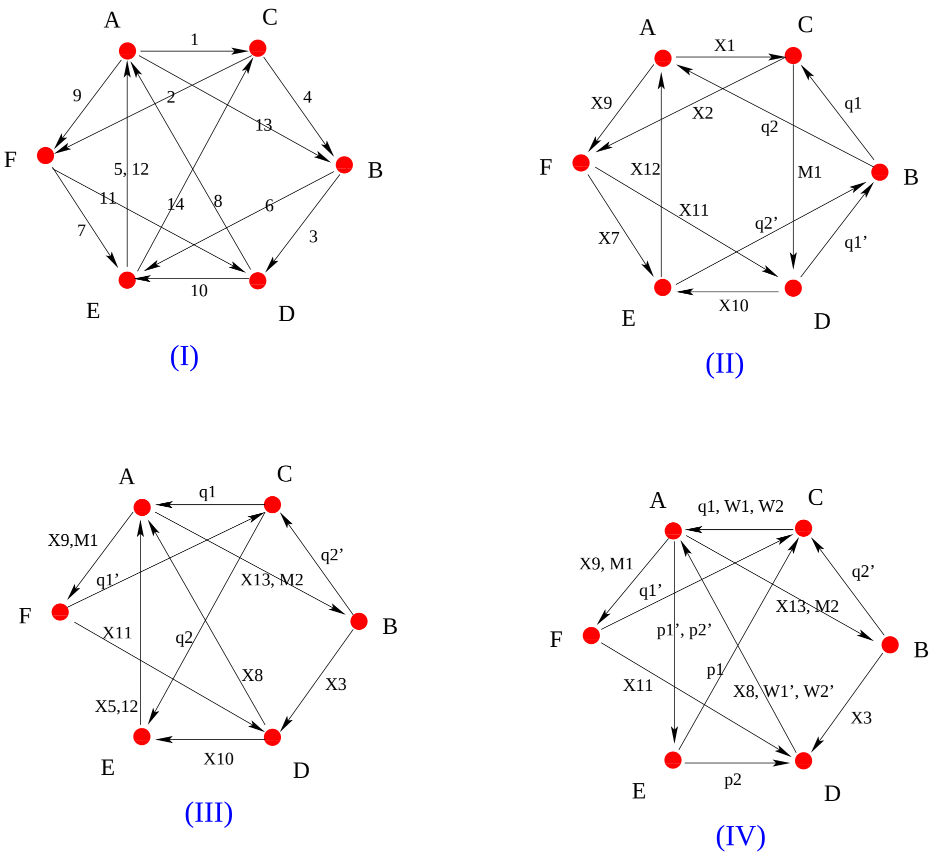

Figure 12: The four Seiberg dual phases of the cone over dP_3. These are respectively isomorphic to cases II, I, II and IV obtained in [9].

Let us derive the superpotential. Before dual transformation, the superpotential is [6]

$$\begin{aligned} W_I = & X_3X_8X_{13} - X_8X_9X_{11} - X_5X_6X_{13} - X_1X_3X_4X_{10}X_{12} + \\ & + X_7X_9X_{12} + X_4X_6X_{14} + X_1X_2X_5X_{10}X_{11} - X_2X_7X_{14} . \end{aligned}$$

After dualization, superpotential is rewritten as

$$\begin{aligned} W' = & M_1'X_8 - X_8X_9X_{11} - X_5M_2' - X_1M_1X_{10}X_{12} + \\ & + X_7X_9X_{12} + M_2X_{14} + X_1X_2X_5X_{10}X_{11} - X_2X_7X_{14} . \end{aligned}$$

It is very clear that fields $X_8, M_1', X_5, M_2', X_{14}, M_2$ are all massive. Furthermore, we need to add the meson part

$$W_{\text{meson}} = M_1q_1'q_1 - M_2q_1q_2' - M_1'q_1'q_2 + M_2'q_2'q_2 ,$$

where we determine the sign as follows: since the term $M_1'X_8$ in W' in positive, we need term $M_1'q_1'q_2$ to be negative. After integration all massive fields, we get the superpotential as

$$W_{\text{II}} = -q_1'q_2X_9X_{11} - X_1M_1X_{10}X_{12} + X_7X_9X_{12} + X_1X_2q_2'q_2X_{10}X_{11} - X_2X_7q_1q_2' + M_1q_1'q_1 .$$

The charge matrix now becomes

$$\begin{array}{cc} & \begin{array}{cccccccccccc} q_1 & q_2 & q'_1 & q'_2 & X_1 & X_2 & X_7 & X_9 & X_{10} & X_{11} & M_1 & X_{12} \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 \end{pmatrix} \end{array} .$$

This is in precise agreement with [28]; very re-assuring indeed!

Without further ado let us present the remaining cases. The charge matrix for the third one (dualising node C of (I)) is

$$\begin{array}{cc} & \begin{array}{cccccccccccccc} q_1 & q'_1 & q'_2 & q_2 & X_5 & X_{12} & X_3 & X_8 & X_9 & M_1 & X_{10} & X_{11} & X_{13} & M_2 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \end{pmatrix} \end{array}$$

with superpotential

$$\begin{aligned} W_{III} = {} & X_3X_8X_{13} - X_8X_9X_{11} - X_5q_2q'_2X_{13} - M_2X_3X_{10}X_{12} + \\ & + q_2q'_1X_9x_{12} + M_1X_5X_{10}X_{11} - M_1q_1q'_1 + M_2q_1q'_2 . \end{aligned}$$

Finally the fourth case (dualising node E of (III)) has the charge matrix

$$\begin{array}{cc} & \begin{array}{cccccccccccccccccc} q_1 & W_1 & W_2 & q'_1 & q'_2 & X_3 & X_8 & W'_1 & W'_2 & X_9 & M_1 & X_{11} & X_{13} & M_2 & p_1 & p'_1 & p'_2 & p_2 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

with superpotential

$$\begin{aligned} W_{IV} = {} & X_3X_8X_{13} - X_8X_9X_{11} - W_1q'_2X_{13} - M_2X_3W'_2 + q'_1X_9W_2 + M_1W'_1X_{11} - \\ & - M_1q_1q'_1 + M_2q_1q'_2 + W_1p_1p'_1 - W_2p_1p'_2 - W'_1p_2p'_1 + W'_2p_2p'_2 . \end{aligned}$$

7. Picard-Lefschetz monodromy and Seiberg duality

In this section let us make some brief comments about Picard-Lefschetz theory and Seiberg duality, a relation between which has been within the literature [29]. It was argued in [30] that at least in the case of D3-branes placed on ADE conifolds [31, 32] Seiberg duality for $\mathcal{N} = 1$ SUSY gauge theories can be geometrised into Picard-Lefschetz monodromy. Moreover in [28] Toric Duality is interpreted as Picard-Lefschetz monodromy action on the 3-cycles.

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On the level of brane setups, this interpretation seems to be reasonable. Indeed, consider a brane crossing process in a brane interval picture. Two branes separated in x^6 approach, are exchanged, and move back. The T-dual operation on the singularity corresponds to choosing a collapsed cycle, decreasing its B-field to zero, and continuing to negative values. This last operation is basically the one generating Picard-Lefschetz monodromy at the level of homology classes. Similarly, the closing and reopening of diamonds corresponds to continuations past infinite coupling of the gauge theories, namely to changes in the T-dual B-fields in the collapsed cycles.

It is the purpose of this section to point out the observation that while for restricted classes of theories the two phenomena are the same, in general Seiberg duality and a naïve application of Picard-Lefschetz (PL) monodromy do not seem to coincide. We leave this issue here as a puzzle, which we shall resolve in an upcoming work.

The organisation is as follows. First we briefly introduce the concept of Picard-Lefschetz monodromy for the convenience of the reader and to establish some notation. Then we give two examples: the first is one with two Seiberg dual theories not related by PL and the second, PL dual theories not related by Seiberg duality.

7.1 Picard-Lefschetz monodromy

We first briefly remind the reader of the key points of the PL theory [33]. Given a singularity on a manifold M and a basis $\{\Delta_i\} \subset H_{n-1}(M)$ for its vanishing $(n-1)$-cycles, going around these vanishing cycles induces a monodromy, acting on arbitrary cycles $a \in H_\bullet(M)$; moreover this action is computable in terms of intersection $a \circ \Delta_i$ of the cycle a with the basis:

Theorem. *The monodromy group of a singularity is generated by the Picard-Lefschetz operators h_i, corresponding to a basis $\{\Delta_i\} \subset H_{n-1}$ of vanishing cycles. In particular for any cycle $a \in H_{n-1}$ (no summation in i)*

$$h_i(a) = a + (-1)^{\frac{n(n+1)}{2}}(a \circ \Delta_i)\Delta_i \,.$$

More concretely, the PL monodromy operator h_i acts as a matrix $(h_i)_{jk}$ on the basis Δ_j:

$$h_i(\Delta_j) = (h_i)_{jk}\Delta_k \,.$$

Next we establish the relationship between this geometric concept and a physical interpretation. According geometric engineering, when a D-brane wraps a vanishing cycle in the basis, it give rise to a simple factor in the product gauge group. Therefore the total number of vanishing cycles gives the number of gauge group factors. Moreover, the rank of each particular factor is determined by how many times it wraps that cycle.

For example, an original theory with gauge group $\prod_j \mathrm{SU}(M_j)$ is represented by the brane wrapping the cycle $\sum_j M_j\Delta_j$. Under PL monodromy, the cycle undergoes the transformation

$$\sum_j M_j\Delta_j \Longrightarrow \sum_j M_j(h_i)_{jk}\Delta_k \,.$$

Physically, the final gauge theory is $\prod_k \mathrm{SU}(\sum_j M_j(h_i)_{jk})$.

The above shows how the rank of the gauge theory changes under PL. To determine the theory completely, we also need to see how the matter content transforms. In geometric engineering, the matter content is given by intersection of these cycles Δ_j. Incidentally, our Inverse Algorithm gives a nice way and alternative method of computing such intersection matrices of cycles.

Let us take $a = \Delta_j$, then

$$h_i(\Delta_j) = \Delta_j + (\Delta_j \circ \Delta_i)\Delta_i\,.$$

This is particularly useful to us because $(\Delta_j \circ \Delta_i)$, as is well-known, is the anti-symmetrised adjacency matrix of the quiver (for a recent discussion on this, see [28]). Indeed this intersection matrix of (the blowup of) the vanishing homological cycles specifies the matter content as prescribed by D-branes wrapping these cycles in the mirror picture. Therefore we have $(\Delta_j \circ \Delta_i) = [a_{ji}] := a_{ji} - a_{ij}$ for $j \neq i$ and for $i = j$, we have the self-intersection numbers $(\Delta_i \circ \Delta_i)$. Hence we can safely write (no summation in i)

$$\Delta_j^{\text{dual}} = h_i(\Delta_j) = \Delta_j + [a_{ji}]\Delta_i \tag{7.1}$$

for a_{ji} the quiver (matter) matrix when Seiberg dualising on the node i; we have also used the notation $[M]$ to mean the antisymmetrisation $M - M^t$ of matrix M. Incidentally in the basis prescribed by $\{\Delta_i\}$, we have the explicit form of the Picard-Lefschetz operators in terms of the quiver matrix (no summation over indices): $(h_i)_{jk} = \delta_{jk} + [a_{ji}]\delta_{ik}$.

From (7.1) we have

$$\begin{aligned}
[a_{jk}^{\text{dual}}] &:= \Delta_j^{\text{dual}} \circ \Delta_k^{\text{dual}} = (\Delta_j + [a_{ji}]\Delta_i) \circ (\Delta_k + [a_{ki}]\Delta_i) \\
&= [a_{jk}] + [a_{ki}][a_{ji}] + [a_{ji}][a_{ik}] + [a_{ji}][a_{ki}]\Delta_i \circ \Delta_i \\
&= [a_{jk}] + c_i[a_{ij}][a_{ki}]\,,
\end{aligned} \tag{7.2}$$

where $c_i := \Delta_i \circ \Delta_i$, are constants depending only on self-intersection.

We observe that our quiver duality rules obtained from field theory (see beginning of section 6) seem to resemble (7.2), i.e. when $c_i = 1$ and $j, k \neq i$. However the precise relation of trying to reproduce Seiberg duality with PL theory still remains elusive.

7.2 Two interesting examples

However the situation is not as simple. In the following we shall argue that while Seiberg duality and a straightforward Picard-Lefschetz transformation certainly do have common features and that in restricted classes of theories such as those in [30], for general singularities the two phenomena may bifurcate.

We first present two theories related by Seiberg duality that cannot be so by Picard-Lefschetz. Consider the standard $\mathbb{C}^3/\mathbb{Z}_3$ theory with $a_{ij} = \left(\begin{smallmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{smallmatrix}\right)$ and gauge group $\mathrm{U}(1)^3$, given in (a) of figure 13. Let us Seiberg-dualise on node A to obtain a theory (b), with matter content $a_{ij}^{\text{dual}} = \left(\begin{smallmatrix} 0 & 3 & 0 \\ 0 & 0 & 6 \\ 3 & 0 & 0 \end{smallmatrix}\right)$ and gauge group $\mathrm{SU}(2) \times \mathrm{U}(1)^2$. Notice especially that the rank of the gauge group factors in part (b) are $(2, 1, 1)$ while those in part (a) are $(1, 1, 1)$. Therefore theory (b) has total rank 4 while (a) has only 3. Since geometrically PL only

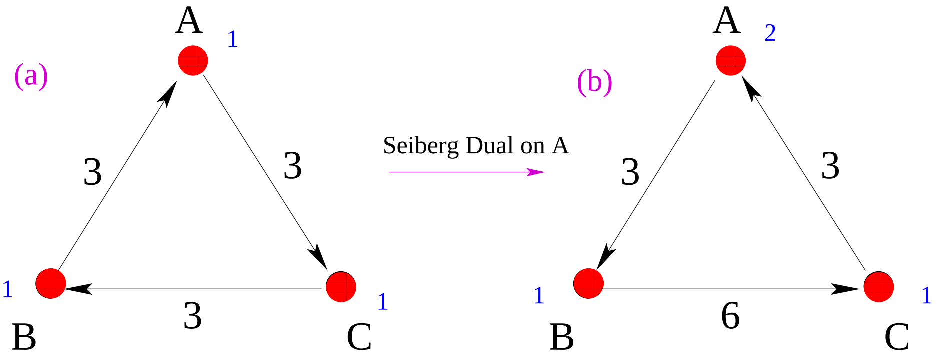

Figure 13: Seiberg Dualisation on node A of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold theory. The subsequent theory cannot be obtained by a Picard-Lefschetz monodromy transformation.

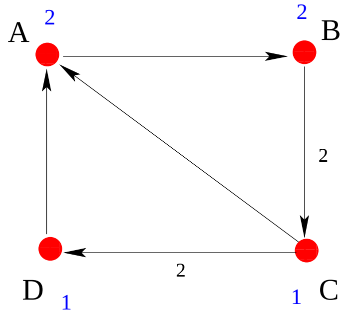

Figure 14: A non-abelian phase of the complex cone over dP_3. This example is Picard-Lefschetz dual to the other 4 examples in figure 12 but not Seiberg dual thereto.

shuffles the vanishing cycles and certainly preserves their number, we see that (a) and (b) cannot be related by PL even though they are Seiberg duals.

On the other hand we give an example in the other direction, namely two Picard-Lefschetz dual theories which are not Seiberg duals. Consider the case given in figure 14, this is a phase of the theory for the complex cone over dP3 as given in [34]. This is PL dual to any of the 4 four phases in figure 12 in the previous section by construction with (p, q)-webs. Note that the total rank remains 6 under PL even though the number of nodes changed. However Seiberg duality on any of the allowed node on any of the 4 phases cannot change the number of nodes. Therefore, this example in figure 14 is not Seiberg dual to the other 4.

What we have learnt in this short section is that Seiberg duality and a naïve application of Picard-Lefschetz monodromy seem to have discrepancies for general singularities. The resolution of this puzzle will be delt with in a forthcoming work.

8. Conclusions

In [6, 8] a mysterious duality between classes of gauge theories on D-branes probing toric singularities was observed. Such a Toric Duality identifies the infrared moduli space of very different theories which are candidates for the world-volume theory on D3-branes at threefold singularities. On the other hand, [11, 17] have recognised certain brane-moves for brane configurations of certain toric singularities as Seiberg duality.

In this paper we take a unified view to the above. Indeed we have provided a physical interpretation for toric duality. The fact that the gauge theories share by definition the same moduli space motivates the proposal that they are indeed physically equivalent in the infrared. In fact, we have shown in detail that toric dual gauge theories are connected by Seiberg duality.

This task has been facilitated by the use of T-dual configurations of NS and D-branes, in particular brane intervals and brane diamonds [23]. These constructions show that the Seiberg duality corresponds in the singularity picture to a change of B-fields in the collapsed cycles. Hence, the specific gauge theory arising on D3-branes at a given singularity, depends not only on the geometry of the singularity, but also on the B-field data. Seiberg duality and brane diamonds provide us with the tools to move around this more difficult piece of the singular moduli space, and probe different phases.

This viewpoint is nicely connected with that in [6, 8], where toric duals were obtained as different partial resolutions of a given orbifold singularity, $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$, leading to equivalent geometries (with toric diagrams equivalent up to unimodular transformations). Specifically, the original orbifold singularity has a specific assignments of B-fields on its collapsed cycles. Different partial resolutions amount to choosing a subset of such cycles, and blowing up the rest. Hence, in general different partial resolutions leading to the same geometric singularity end up with different assignments of B-fields. This explains why different gauge theories, related by Seiberg duality, arise by different partial resolutions.

In particular we have examined in detail the toric dual theories for the generalised conifold $xy = z^2w^2$, the partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ exemplified by the complex cones over the zeroth Hirzebruch surface as well as the second del Pezzo surface. We have shown how these theories are equivalent under the above scheme by explicitly having

1. unimodularly equivalent toric data;
2. the matter content and superpotential related by Seiberg duality;
3. the T-dual brane setups related by brane-crossing and diamond duality.

The point d'appui of this work is to show that the above three phenomena are the same.

As a nice bonus, the physical understanding of toric duality has allowed us to construct new toric duals in cases where the partial resolution technique provided only one phase. Indeed the exponential running-time of the Inverse Algorithm currently prohibits larger embeddings and partial resolutions. Our new perspective greatly facilitates the calculation of new phases. As an example we have constructed three new phases for the cone over del Pezzo three one of which is in reassuring agreement with a recent work [28] obtained from completely different methods.

Another important direction is to understand the physical meaning of Picard-Lefschetz transformations. As we have pointed out in section 7, PL transformation and Seiberg

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duality are really two different concepts even though they coincide for certain restricted classes of theories. We have provided examples of two theories which are related by one but not the other. Indeed we must pause to question ourselves. For those which are Seiberg dual but not PL related, what geometrical action does correspond to the field theory transformation. On the other hand, perhaps more importantly, for those related to each other by PL transformation but not by Seiberg duality, what kind of duality is realized in the dynamics of field theory? Does there exists a new kind of dynamical duality not yet uncovered??

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